

# Langlands Conjectures: Functoriality

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## 1 Introduction

These are notes from a talk on the Langlands conjectures. While I believe everything written here to be spiritually correct, there are many details that I'm not sure of, so statements in these notes should be checked before being repeated.

For simplicity, all reductive groups are connected.

There are two main sides to the Langlands conjectures, namely reciprocity and functoriality. We've seen the reciprocity conjectures for  $GL_n$ ; we'll state the reciprocity conjectures for general reductive groups, as well as the functoriality conjectures. They are closely related, and though often stated separately, one should understand them as inseparable parts of the Langlands philosophy.

In the unramified case there is a nice classification of the representations on both sides of the Langlands correspondence, so the conjectures are well understood, and it gives some small insight into why one would expect something like Langlands to be true. So we'll spend some time trying to understand unramified representations, and this will form the basis for our understanding of the Langlands conjectures.

To begin let's recall the objects we're working with, and the statements of local and global Langlands for  $GL_n$ . Throughout,  $k$  will denote a non-archimedean local field and  $K$  will denote a global field; if the field is local or global, we'll use  $K$ .

### 1.1 Local Langlands for $GL_n$

We'll discuss local Langlands only in the non-archimedean case (there are also conjectures in the archimedean case, which we'll ignore).

Let  $k$  be a non-archimedean local field; the local objects involved in the Langlands correspondence are these. We say a representation  $V$  of  $GL_n(k)$  is *admissible* if for every compact open subgroup  $H \subset GL_n(k)$  the subspace of fixed vectors  $V^H$  is finite-dimensional, and  $V = \bigcup_{H \text{ compact open}} V^H$ . Essentially, admissible representations are built out of finite-dimensional pieces. To such a thing we can associate a local  $L$ -factor and  $\varepsilon$ -factor, as well as  $L$ - and  $\varepsilon$ -factors for pairs.

Recall the exact sequence

$$1 \rightarrow I_k \rightarrow \text{Gal}(\bar{k}/k) \rightarrow \text{Gal}(\bar{\kappa}/\kappa) \rightarrow 1,$$

where  $\kappa$  is the residue field of  $k$ . The *Weil group* of  $k$  is the subgroup of  $\text{Gal}(\bar{k}/k)$  mapping to integer powers of  $\text{Frob}_p \in \text{Gal}(\bar{\kappa}/\kappa)$ , i.e. the preimage of  $\mathbb{Z} \subset \hat{\mathbb{Z}} = \text{Gal}(\bar{\kappa}/\kappa)$ . The *Weil-Deligne group* is  $W'_k := W_k \ltimes \mathbb{G}_a$  with respect to the action  $w x w^{-1} = \|w\| x$ . We'll consider representations of the Weil-Deligne group, which can also be understood as representations of the Weil group

together with a nilpotent endomorphism  $N$ . We can also associate  $L$ - and  $\varepsilon$ -factors to Weil-Deligne representations.

The local Langlands conjectures for  $\mathrm{GL}_n$  essentially predict a correspondence between admissible representations of  $\mathrm{GL}_n(k)$  and  $n$ -dimensional Weil-Deligne representations.

There is an equivalence of categories between bounded Weil-Deligne representations and  $\ell$ -adic Galois representations (this is Grothendieck's monodromy theorem), so local Langlands also associates admissible representations to  $\ell$ -adic Galois representations.

**Conjecture 1** (Local Langlands for  $\mathrm{GL}_n$ ). *Let  $k$  be a non-archimedean local field. For each  $n \geq 1$  there is a (unique) bijection*

$$\left\{ \begin{array}{l} \text{(isom. classes of) irreducible admissible} \\ \text{complex representations of } \mathrm{GL}_n(k) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{(isom. classes of) } F\text{-semisimple representa-} \\ \text{tions } W'_k \rightarrow \mathrm{GL}_n(\mathbb{C}) \end{array} \right\}$$

$$\pi \longleftrightarrow \rho_\pi$$

satisfying the following conditions.

1. The  $n = 1$  case is local class field theory.
2. If  $\pi, \pi'$  are associated to  $\rho_\pi, \rho_{\pi'}$  then

$$L(s, \pi \times \pi') = L(s, \rho_\pi \otimes \rho_{\pi'})$$

and

$$\varepsilon(s, \pi \times \pi', \psi) = \varepsilon(s, \rho_\pi \otimes \rho_{\pi'}, \psi)$$

where the left hand sides are  $L$ - and  $\varepsilon$ -factors for pairs.

3. If  $\chi : k^\times \rightarrow \mathbb{C}^\times$  corresponds by local class field theory to  $\chi : W_k \rightarrow \mathbb{C}^\times$  then  $\rho_{\pi \otimes \chi} = \rho_\pi \otimes \chi$ .
4. If  $\pi$  has central character  $\chi : k^\times \rightarrow \mathbb{C}^\times$  then (again writing  $\chi$  for its local class field theory partner)  $\det \rho_\pi = \chi$ .
5.  $\rho_{\pi^\vee} = \rho_\pi^\vee$ .

Local Langlands for  $\mathrm{GL}_n$  is now a theorem; see e.g. [HT01] for more details.

## 1.2 Global Langlands for $\mathrm{GL}_n$

Now let  $K$  be a global field, and  $\mathbb{A}_K$  its ring of adeles. An automorphic representation of  $\mathrm{GL}_n(\mathbb{A}_K)$  is essentially a constituent of the regular representation

$$L^2(\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K)).$$

The essential fact to remember is that every automorphic representation  $\pi$  of  $\mathrm{GL}_n(\mathbb{A}_K)$  decomposes as a restricted tensor product  $\pi = \otimes' \pi_v$  into admissible representations  $\pi_v$  of  $\mathrm{GL}_n(K_v)$  at each place  $v$  of  $K$  (the Flath decomposition). Intuitively, modding by  $\mathrm{GL}_n(K)$  in the definition means that being automorphic is a strong compatibility condition between the local representations. We can define an  $L$ -function

$$L(s, \pi) = \prod_v L(s, \pi_v)$$

as product of local  $L$ -factors, and similarly  $\varepsilon$ -factors, as well as  $L$ -functions and  $\varepsilon$ -factors for pairs.

On the other side of the global Langlands correspondance is the conjectural Langlands group  $\mathcal{L}_K$ , a global version of the Weil-Deligne group. Supposing such a group exists, the global Langlands conjecture for  $\mathrm{GL}_n$  predicts a correspondence between automorphic representations of  $\mathrm{GL}_n(\mathbb{A}_K)$  and admissible representations of  $\mathcal{L}_K$ . The precise form this conjecture should take is not known, which is understandable since the Langlands group is not even known to exist.

However, as in the local case, the admissible representations of  $\mathcal{L}_K$  should include Galois representations, and if we narrow down the automorphic side we can get a concrete conjecture of a correspondence between automorphic representations and Galois representations.

First, we'll need some definitions. Recall that an automorphic representation is *cuspidal* if it occurs in a smaller space  $L_0^2(\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K))$  defined by the vanishing of some integral. There is also a notion of  $L$ -algebraic, which concerns the infinite part; essentially, at the infinite place there's a complex space of parameters that can appear (classifying the "infinitesimal character", which is the character by which the center of the universal enveloping algebra acts), and a representation is  $L$ -algebraic if these parameters at infinite places are actually integers.

An  $\ell$ -adic Galois representation is *geometric* if it is unramified at almost all places and de Rham at  $\ell$  (de Rham is a technical condition from  $p$ -adic Hodge theory). A collection of  $\ell$ -adic Galois representations for each  $\ell$  is *compatible* if it satisfies a compatibility condition we won't go into; but for example, from an elliptic curve  $E$  (or abelian variety) we get an  $\ell$ -adic Galois representation from the Tate module  $T_\ell E$  for each  $\ell$ , and these form a compatible system. Also, given a compatible system of  $\ell$ -adic representations, we get a collection of local Weil-Deligne representations by restricting to a local Galois representation  $\mathrm{Gal}(\overline{K}_v/K_v) \rightarrow \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{GL}_n(\mathbb{Q}_\ell)$  and then using Grothendieck's monodromy theorem to produce a Weil-Deligne representation (the compatibility ensures that this is independent of  $\ell$ ).

**Conjecture 2** (Global Langlands for  $\mathrm{GL}_n$ ). *Let  $K$  be a global field. For each  $n \geq 1$  there is a (unique) bijection*

$$\left\{ \begin{array}{l} \text{(isom. classes of) } L\text{-algebraic cuspidal au-} \\ \text{tomorphic representations of } \mathrm{GL}_n(\mathbb{A}_K) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{(isom. classes of) compatible systems of ir-} \\ \text{reducible geometric } \ell\text{-adic representations} \\ \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \end{array} \right\}$$

$$\pi \longleftrightarrow \{\rho_{\pi, \ell}\}$$

*such that the local admissible representations on the left hand side correspond, by local Langlands, to the local Weil-Deligne representations on the right hand side.*

For more details see e.g. [Tay04] and [BG14].

## 2 Unramified Representations and the $L$ -group

### 2.1 Unramified Representations of $\mathrm{GL}_n$

Back to the local case, with  $k$  a (non-archimedean) local field. A Weil-Deligne representation (regarded as a pair  $(\rho, N)$  of a Weil representation  $\rho$  and endomorphism  $N$ ) is *unramified* if  $N = 0$  and  $\rho(I_k) = 1$ ; that is, we have a factorization  $\rho : W_k \rightarrow \mathbb{Z} \rightarrow \mathrm{GL}_n(\mathbb{C})$ . So an unramified Weil-Deligne representation is completely determined by the image of Frob. This image is a semisimple element of  $\mathrm{GL}_n(\mathbb{C})$ , determined up to conjugacy.

By local Langlands, these correspond to a class of admissible representations of  $\mathrm{GL}_n(k)$ . We say an admissible representation of  $\mathrm{GL}_n(k)$  is *unramified* if there is a fixed vector for the (maximal compact) subgroup  $\mathrm{GL}_n(\mathcal{O}_k)$ . This definition might take some unraveling to see why it

deserves the name unramified, but suffice it to say that these are the admissible representations corresponding to unramified Weil-Deligne representations.

In fact we can see directly that these representations are also parametrized by semisimple conjugacy classes in  $\mathrm{GL}_n(\mathbb{C})$ . This requires a fair amount of theory, but we can sketch the ideas.

Let  $B \subset \mathrm{GL}_n(k)$  be the Borel subgroup of upper triangular matrices. Given any  $n$ -tuple of complex numbers  $u = (u_1, \dots, u_n)$ , we can define a 1-dimensional complex representation of  $B$  by

$$\chi_u(b_{ij}) = |b_{11}|^{u_1 + \frac{n-1}{2}} |b_{22}|^{u_2 + \frac{n-3}{2}} \cdots |b_{nn}|^{u_n + \frac{1-n}{2}}.$$

(The extra factors in the exponents are a technical detail that can be ignored). Then we produce a representation of  $\mathrm{GL}_n(k)$  by induction. (This is an example of the very important method of *parabolic induction*, which is essential for understanding admissible representations). It is a fact that this induced representation has a unique irreducible unramified constituent  $\pi_u$ . Furthermore, every irreducible unramified representation of  $\mathrm{GL}_n(k)$  arises in this way.

This is not quite a classification yet. It turns out that two tuples  $u, u'$  define the same representation precisely when

$$(u'_1, \dots, u'_n) \equiv (u_{\sigma(1)}, \dots, u_{\sigma(n)}) \bmod \left( \frac{2\pi i}{\log p} \right) \mathbb{Z}^n$$

for some permutation  $\sigma \in S_n$ ; or equivalently, when

$$\begin{pmatrix} p^{-u'_1} & & 0 \\ & \ddots & \\ 0 & & p^{-u'_n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p^{-u_1} & & 0 \\ & \ddots & \\ 0 & & p^{-u_n} \end{pmatrix}$$

are conjugate in  $\mathrm{GL}_n(\mathbb{C})$  (exponentiating accounts for equivalence mod  $\frac{2\pi i}{\log p} \mathbb{Z}^n$ , and conjugacy accounts for permutations). We see that an irreducible unramified representation  $\pi$  of  $\mathrm{GL}_n(k)$  is classified by a conjugacy class of semisimple elements in  $\mathrm{GL}_n(\mathbb{C})$ , which we denote by  $c(\pi)$ .

So the local Langlands correspondence for  $\mathrm{GL}_n$  is (more) easily understood in the unramified case: both sides are parametrized by semisimple conjugacy classes in  $\mathrm{GL}_n(\mathbb{C})$ , and this gives the desired bijection.

This parametrization by semisimple conjugacy classes is called the Satake classification, and it extends to general reductive groups. The unramified representations of  $G(k)$  for a split reductive group  $G$  are classified by semisimple conjugacy classes in the dual group  $\hat{G}$ ; for non-split groups some extra Galois data is required. The classification in general is captured by the  $L$ -group, which we now discuss.

## 2.2 The $L$ -Group

Let's briefly recall the classification of reductive groups by root data. That root data classify reductive groups (over an algebraically closed field) is a remarkable fact, and even rather mysterious—at least to me. But let's try to motivate it a bit.

Let  $G$  be a connected reductive group over an algebraically closed field. The starting point that seems the least mysterious to me is the Bruhat decomposition, which says that if  $B \supset T$  are a Borel subgroup and maximal torus of  $G$  (which we call a *Borel pair*), and  $W = N_G(T)/T$  the associated Weyl group, then

$$G = \coprod_{w \in W} BwB,$$

and there is a formula for how these double cosets multiply. Thus the group  $G$  is essentially encoded in the data of a Borel pair  $B \supset T$  and the Weyl group of  $T$ .

As  $T$  is a split torus (our field is algebraically closed), we can remember its structure simply from the character group  $X^*(T)$ , or its dual, the cocharacter group  $X_*(T)$ . To remember also the Borel and Weyl group, we add the data of “simple roots”  $\Delta^* \subset X^*(T)$  and “simple coroots”  $\Delta_* \subset X_*(T)$ . The “roots” are the (non-trivial) characters of  $T$  that appear in the action of  $T$  on  $\text{Lie}(G)$ , and “simple roots” are a certain subset of these depending on the Borel  $B$ . All this together allows us to encode the group  $G$  in the combinatorial data  $\Psi(G) = (X^*(T), \Delta^*, X_*(T), \Delta_*)$ .

Abstractly, a *root datum* is a tuple  $(X^*, R^*, X_*, R_*)$  where  $X^*, X_*$  are finite-rank free  $\mathbb{Z}$ -modules dual under a pairing  $\langle \cdot, \cdot \rangle : X^* \times X_* \rightarrow \mathbb{Z}$ , and  $R^* \subset X^*, R_* \subset X_*$  are finite subsets with a bijection  $R^* \leftrightarrow R_*, \alpha \leftrightarrow \alpha^\vee$  satisfying  $\langle \alpha, \alpha^\vee \rangle = 2$ ; furthermore, the reflections

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$$

are required to preserve  $R^*$  and  $R_*$  respectively. (The group generated by the reflections  $s_\alpha$  is the Weyl group). A *based root datum*  $(X^*, \Delta^*, X_*, \Delta_*)$  is obtained by choosing a set of “simple roots”  $\Delta^* \subset R^*$  and the dual coroots  $\Delta_* \subset R_*$ . (It is *reduced* if no root is twice another root).

**Theorem 3.** *The map*

$$\left\{ \begin{array}{l} \text{(isom. classes of) connected reductive} \\ \text{groups over an algebraically closed field} \end{array} \right\} \longrightarrow \{ \text{(isom. classes of) reduced based root data} \}$$

$$G \longmapsto (X^*(T), \Delta^*, X_*(T), \Delta_*)$$

*is a bijection. Furthermore, an isomorphism of reductive groups is determined up to inner automorphism by the induced map on root data.*

There is a natural notion of duality for (based) root data, given by exchanging characters for cocharacters and roots for coroots:

$$(X^*, \Delta^*, X_*, \Delta_*)^\vee = (X_*, \Delta_*, X^*, \Delta^*).$$

By the above theorem, this gives a duality for reductive groups. A *dual group* for  $G$  is a complex reductive Lie group  $\hat{G}$  equipped with an isomorphism  $\Psi(\hat{G}) \cong \Psi(G)^\vee$ . We can make this definition for a connected reductive group over any field, but note that  $\hat{G}$  depends only on the isomorphism class over an algebraic closure, i.e. forms of the same group have the same dual.

Now suppose  $G$  is defined over a local or global field  $K$ . The classification theorem implies that

$$1 \rightarrow G^{\text{ad}} \rightarrow \text{Aut}(\hat{G}) \rightarrow \text{Aut}(\Psi(\hat{G})) \rightarrow 1$$

is a split exact sequence. The Galois group  $\text{Gal}(\bar{K}/K)$  acts on  $\Psi(\hat{G}) = \Psi(G)^\vee$ , and a choice of splitting for the above exact sequence produces an action of  $\text{Gal}(\bar{K}/K)$  on  $\hat{G}$ , which factors through  $\text{Gal}(K'/K)$  if  $G$  is split over  $K'$ . We can also consider this as an action of the Weil group  $W_K$  by composing with the natural morphism  $W_K \rightarrow \text{Gal}(\bar{K}/K)$ . (There is a Weil group for global fields, which we haven’t discussed). This action remembers  $G$  up to inner automorphism.

The *L-group* or *Langlands dual group* of a reductive group  $G$  defined over a local or global field  $k$  is

$${}^L G = \hat{G} \rtimes W_K,$$

the semidirect product taken with respect to the above action. In particular, if  $G$  is split over  $K$ , the  ${}^L G = \hat{G} \rtimes W_K$ . Since the action only remembers  $G$  up to inner automorphism, inner forms have the same  $L$ -group.

We could just as well define

$${}^L G = \hat{G} \rtimes \text{Gal}(\bar{K}/K),$$

or even

$${}^L G = \hat{G} \rtimes \text{Gal}(L/K),$$

for some sufficiently large extension  $L/K$ , and everything we're going to do would still work. The main difference is that the Weil group has more representations than the Galois group, so the Weil form of the  $L$ -group admits more  $L$ -homomorphisms than the Galois form (a notion which we'll define later). So the Weil form is more general in some sense, but at times it's more useful or illuminating to use the Galois form.

For more discussion on the  $L$ -group, see [BR94] and [Cog03].

### 2.3 The Satake Classification

Let  $G$  be a reductive group defined over a local field  $k$ . We can define a notion of *unramified* representation of  $G$  whenever  $G$  is *unramified*, which is when  $G$  is quasisplit over  $\mathbb{Q}_p$  (meaning  $G$  has a Borel subgroup and maximal torus  $B \supset T$  defined over  $\mathbb{Q}_p$ ) and split over an unramified extension  $E/\mathbb{Q}_p$ . The Satake classification states that unramified representations  $\pi$  of  $G(k)$  are classified by conjugacy classes  $c(\pi)$  in  ${}^L G$  whose projection to  $\hat{G}$  is semisimple and whose projection to  $W_k$  is Frob. If  $G(k) = G(K_v)$  is a local form at a place  $v$  of a group defined over a global field  $K$ , and if  $G$  is split over a field in which  $v$  is unramified, this is the same as a conjugacy class in the global  $L$ -group  ${}^L G = \hat{G} \rtimes \text{Gal}(\bar{K}/K)$  whose projection to  $\hat{G}$  is semisimple and whose projection to  $\text{Gal}(\bar{K}/K)$  is  $\text{Frob}_v$ . The conjugacy class  $c(\pi)$  is called a *Satake parameter*.

Now suppose  $G$  is defined over a global field  $K$ , split over an extension  $E/K$ , and unramified at almost all places. Let  $\pi$  be an automorphic representation of  $G(\mathbb{A}_K)$ , which in the decomposition  $\pi = \otimes' \pi_v$  is unramified at almost all places. To  $\pi$  we can associate a set of conjugacy classes  $c(\pi) = \{c(\pi_v)\}$  in  $\hat{G} \rtimes \text{Gal}(E/K)$ , the Satake parameters at all unramified places.

The point is that the automorphic representation  $\pi$  is encoded in the much more concrete data of its Satake parameters. For further discussion of Satake parameters, see [Art03].

## 3 Langlands for Reductive Groups

The Langlands conjectures in their full generality encompass all reductive groups, not just  $\text{GL}_n$ . Reductive is a good class of groups to consider because we're essentially studying their representations, and reductive groups have good representation theory. For example, an algebraic group in characteristic zero is reductive precisely when its category of representations is semisimple. One can consider automorphic representations of algebraic groups that are not reductive, but it's not clear if they are connected to other mathematics (in particular, it's not clear if they're connected to Galois theoretic objects).

In order to start talking about this, we need to make some more general definitions. Many of them are the same as in the case of  $\text{GL}_n$ , but perhaps it's reassuring to go over them again.

In the local case, let  $k$  be a local field and  $G$  a reductive group over  $k$ . A representation  $V$  of  $G(k)$  is *admissible* if for any compact open subgroup  $H \subset G(k)$  the subspace of fixed vectors  $V^H$  is finite-dimensional, and  $V = \bigcup_{H \text{ compact open}} V^H$ . These should have local  $L$ - and  $\varepsilon$ -factors associated to them, but we'll return to this later.

In the global case, let  $K$  be a global field and  $G$  a reductive group over  $K$ . An *automorphic representation* of  $G(\mathbb{A}_K)$  is essentially a constituent of

$$L^2(G(K) \backslash G(\mathbb{A}_K)).$$

As in the  $\mathrm{GL}_n$  case, these decompose as a restricted tensor product  $\pi = \otimes \pi_v$  of admissible representations  $\pi_v$  of  $G(K_v)$ .

To formulate Langlands reciprocity for general reductive groups,  $\mathrm{GL}_n$  is replaced on one side by  $G$  and on the other by  ${}^L G$ . This is not all that changes; there are some aspects of the  $\mathrm{GL}_n$  case that are deceptively simple. Perhaps the biggest is the existence of  $L$ -packets. For general groups, the map from automorphic representations to Galois representations is no longer expected to be a bijection, but a finite-to-one map, and the fibers are called  $L$ -packets.

### 3.1 Local Langlands

In the local case for  $\mathrm{GL}_n$ , the Galois side is F-semisimple representations  $W'_k \rightarrow \mathrm{GL}_n(\mathbb{C})$ . For a general reductive group  $G$  over a local field  $k$ , we say a homomorphism  $\phi : W'_k \rightarrow {}^L G$  is *admissible* if

1. the induced map  $W_k \rightarrow W_k$  is the identity;
2.  $\phi$  is continuous, maps semisimple elements to semisimple elements, and  $\phi(\mathbb{G}_a)$  is unipotent in  $\hat{G}$ ; and
3. if the image of  $\phi$  is contained in a Levi subgroup of a proper parabolic subgroup  $P$  of  ${}^L G$ , then  $P$  is relevant.

For definitions here that I've left out, and more details, see [Cog03].

**Conjecture 4** (Local Langlands). *Let  $k$  be a local field. Then there is a (unique) surjective map*

$$\left\{ \begin{array}{l} \text{(isom. classes of) irreducible admissible} \\ \text{complex representations of } G(k) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{(isom. classes of) admissible homomor-} \\ \text{phisms } W'_k \rightarrow {}^L G \end{array} \right\}$$

*with finite fibers, satisfying the following conditions. Let  $\mathcal{A}_\phi$  be the fiber, i.e.  $L$ -packet, over an admissible homomorphism  $\phi$ .*

1. *If  $\pi \in \mathcal{A}_\phi$ , the central character of  $\pi$  is constructed from  $\phi$  in a specified way.*
2. *Compatibility with twisting: (if  $\alpha \in H^1(W'_K, \mathbb{C}(\hat{G}))$  and  $\chi_\alpha$  is the associated character of  $G(K)$ , then  $\mathcal{A}_{\alpha \cdot \phi} = \mathcal{A}_\phi \cdot \chi_\alpha$ ).*
3. *Some  $\pi \in \mathcal{A}_\phi$  is square integrable modulo  $C(G)$  iff all  $\pi \in \mathcal{A}_\phi$  are square-integrable modulo  $C(G)$  iff  $\phi(W'_K)$  does not lie in a proper Levi subgroup of  ${}^L G$ .*
4. *Some  $\pi \in \mathcal{A}_\phi$  is tempered iff all  $\pi \in \mathcal{A}_\phi$  are tempered iff  $\phi(W_K)$  is bounded.*
5. *If  $H$  is a connected reductive  $K$ -group and  $\eta : H(K) \rightarrow G(K)$  is a  $K$ -morphism with commutative kernel and cokernel, then there is a required compatibility between decompositions for admissible representations of  $G(K)$  and  $H(K)$ . To be precise,  $\eta$  induces a map  ${}^L \eta : {}^L G \rightarrow {}^L H$ , and if we define  $\phi' = {}^L \eta \circ \phi$  for  $\phi : W'_K \rightarrow {}^L G$ , then any  $\pi \in \mathcal{A}_\phi$  regarded as a representation of  $H(K)$ , should decompose into a direct sum of finitely many members of  $\mathcal{A}_{\phi'}$ .*

### 3.2 Global Langlands

The global Langlands conjecture for a general reductive group  $G$  over a global field  $K$  should be a map

$$\left\{ \begin{array}{l} \text{(isom. classes of) irreducible automor-} \\ \text{phic representations of } G(\mathbb{A}_K) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{(isom. classes of) admissible homo-} \\ \text{morphisms } \mathcal{L}_K \rightarrow {}^L G \end{array} \right\}$$

satisfying certain conditions, including a local-global compatibility; but the exact form and specifications are not well understood. There is also not a well-formulated conjecture relating automorphic representations to Galois representations, although there had been some work in this direction. For example, Buzzard and Gee [BG14] give a precise conjecture of a map from certain automorphic representations to Galois representations in the general case.

## 4 Functoriality

Now we turn our attention to functoriality. For more discussion, see [Cog03] and [Art03]. While the reciprocity conjectures relate automorphic representations to Galois representations, functoriality relates automorphic representations of different groups.

If  $H, G$  are connected reductive groups over a (local or global) field  $K$ , an *L-homomorphism* between  ${}^L H$  and  ${}^L G$  is a continuous homomorphism  ${}^L H \rightarrow {}^L G$  such that the induced map  $\hat{H} \rightarrow \hat{G}$  is a morphism of complex Lie groups and the induced map  $W_K \rightarrow W_K$  is the identity.

The principle of functoriality is that *L-homomorphisms* should give rise to maps between admissible/automorphic representations of the groups involved; or more precisely, between *L-packets*. There are certain cases where we can lift individual representations, but there is reason to believe that the lifting should not descend to individual representations in general.

I know two main ways to contextualize this. First, in relation to the Satake classification of unramified representations. Recall that Satake parameters are essentially conjugacy classes in the *L-group*, and they parametrize irreducible unramified representations of the local group. An *L-homomorphism*  ${}^L H \rightarrow {}^L G$  is essentially a recipe for transferring Satake parameters from  $H$  to  $G$ , and so there should be a corresponding transfer of representations from  $H$  to  $G$ .

Second, in relation to Langlands reciprocity. If  $\eta : {}^L H \rightarrow {}^L G$  is an *L-homomorphism*, then composing with  $\eta$  gives a map

$$\left\{ \begin{array}{l} \text{(isom. classes of) admissible homo-} \\ \text{morphisms } W'_k \rightarrow {}^L H \text{ or } \mathcal{L}_K \rightarrow {}^L H \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{(isom. classes of) admissible homo-} \\ \text{morphisms } W'_k \rightarrow {}^L G \text{ or } \mathcal{L}_K \rightarrow {}^L G \end{array} \right\}.$$

Now if these two sets parametrize admissible/automorphic representations of  $H$  or  $G$  respectively, then we get a map of *L-packets* from  $H$  to  $G$ , which is the functorial lift.

Functoriality also implies some cases of reciprocity. If we take  $H = 1$  and  $G = \mathrm{GL}_n$ , then  ${}^L H = \mathrm{Gal}(\bar{K}/K)$ , and an *L-homomorphism*  ${}^L H \rightarrow {}^L G$  is a complex Galois representation. In the global case, this should give rise to a map from automorphic representations of  $H(\mathbb{A}_K)$  to automorphic representations of  $G(\mathbb{A}_K)$ . But the former set has one element, so really we're associating an automorphic representation to a Galois representation.

### 4.1 Local Functoriality

**Conjecture 5** (Local Functoriality). *Let  $k$  be a local field, and  $G, H$  reductive groups over  $k$ . If  $\phi : {}^L H \rightarrow {}^L G$  is an *L-homomorphism*, then there is an associated map*

$$\left\{ \begin{array}{l} \text{(L-packets of) admissible representations} \\ \text{of } H(k) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{(L-packets of) admissible representations} \\ \text{of } G(k) \end{array} \right\}$$

$$\pi \longrightarrow \pi'$$

*called a transfer or lifting (or various other things), which in the unramified case is given by  $c(\pi') = \phi(c(\pi))$ .*



There is no good idea of what should determine the lifting in general. Compatibility with local reciprocity does determine it, but one would hope for a more explicit condition.

Because the functorial lift can be constructed via the reciprocity maps, functoriality is well understood whenever the corresponding case of reciprocity is. Namely,

1.  $k$  archimedean,  $H$  a connected reductive group and  $G$  a quasisplit reductive group;
2.  $k$  non-archimedean,  $H$  and  $G$  general linear groups;
3. and in the unramified case. Suppose  $k$  is a non-archimedean field and  $G, H$  are both unramified over  $k$ . Let  $\pi$  be an unramified representation of  $H(k)$  and  $\phi : W'_k \rightarrow {}^L H$  the corresponding unramified Weil-Deligne representation. Then for any  $L$ -homomorphism  $u : {}^L H \rightarrow {}^L G$ , the composition  $u \circ \phi : W'_k \rightarrow {}^L G$  is unramified. Furthermore the  $L$ -packet for  $u \circ \phi$  contains a unique representation  $\Pi$  of  $G(k)$  which is unramified. This  $\Pi$  is called the *natural unramified lift* of  $\pi$ .

## 4.2 Global Functoriality

**Conjecture 6** (Global Functoriality). *Let  $K$  be a global field, and  $G, H$  reductive groups over  $K$ . If  $\phi : {}^L H \rightarrow {}^L G$  is an  $L$ -homomorphism, then there is an associated lifting*

$$\left\{ \begin{array}{l} (L\text{-packets of}) \\ \text{automorphic representa-} \\ \text{tions of } H(\mathbb{A}_K) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} (L\text{-packets of}) \\ \text{automorphic representa-} \\ \text{tions of } G(\mathbb{A}_K) \end{array} \right\}$$

$$\pi \longrightarrow \pi'$$

such that  $c(\pi'_p) = \phi(c(\pi_p))$  at all places where both are unramified.

The global case is more difficult, because global reciprocity is less well understood. However, we do have local-global compatibility at our disposal. If  ${}^L H \rightarrow {}^L G$  is a global  $L$ -homomorphism, it induces local  $L$ -homomorphisms  ${}^L H_v \rightarrow {}^L G_v$  at all places  $v$ . If  $\pi = \otimes' \pi_v$  is an automorphic representation of  $H(\mathbb{A}_K)$ , then for infinite places  $v$  we know how to lift (as long as  $G$  is quasisplit), and for almost all finite places  $v$  the groups  $G, H$  are both unramified at  $v$ , and so we can take the natural unramified lift  $\Pi_v$  of  $\pi_v$ .

We say that an automorphic representation  $\Pi = \otimes' \Pi_v$  of  $G$  is a *weak functorial lift* of  $\pi = \otimes' \pi_v$  if  $\Pi_v$  is a local functorial lift of  $\pi_v$  at all infinite places and almost all finite places. If it is a local functorial lift at all places, we say  $\Pi$  is a *strong functorial lift* of  $\pi$ .

(Note: we're glossing over the fact noted earlier that in general functoriality operates on the level of  $L$ -packets).

## 5 Examples

There are many specific examples of functoriality that have been proven and made use of. For more discussion of this, including what we cover, see [Cog03].

### 5.1 Automorphic Tensor Product

We can see a hint of functoriality in the form of  $L$ -functions. Namely, to two automorphic representations  $\pi, \pi'$  of  $\mathrm{GL}_n$  there is associated an  $L$ -function  $L(s, \pi \times \pi')$  with nice analytic properties (i.e. analytic continuation and functional equation), which should correspond by Langlands to an  $L$ -function  $L(s, \rho_\pi \otimes \rho_{\pi'})$ .

At the moment, there is no definition for a *representation*  $\pi \times \pi'$ , only its  $L$ -function. But such a representation is predicted by functoriality. If we set

$$H = \mathrm{GL}_m \times \mathrm{GL}_n \quad \text{and} \quad G = \mathrm{GL}_{mn},$$

then

$${}^L H = \mathrm{GL}_m \times \mathrm{GL}_n \times W_K \quad \text{and} \quad {}^L G = \mathrm{GL}_{mn} \times W_K,$$

and the tensor product map  $\mathrm{GL}_m \times \mathrm{GL}_n \rightarrow \mathrm{GL}_{mn}$  extends to an  $L$ -homomorphism  ${}^L H \rightarrow {}^L G$ . The corresponding map of automorphic representations predicted by functoriality is our automorphic tensor product  $\pi, \pi' \mapsto \pi \times \pi'$ .

## 5.2 Lifting Between Inner Forms

We observed earlier that inner forms have the same  $L$ -group. If  $H, G$  are inner forms, then the identity map  ${}^L H \rightarrow {}^L G$  is an  $L$ -homomorphism, and functoriality predicts a map from automorphic representations of  $H$  to  $G$ . In the case where  $H = \mathrm{GL}_2$  and  $G = D^\times$  is the multiplicative group of a rank 2 division algebra over  $K$  (an inner form of  $\mathrm{GL}_2$ ), this is the Jacquet-Langlands correspondence, which was one of the first known cases of functoriality.

## 5.3 Base Change

An important example of functoriality for arithmetic applications is base change. If  $L/K$  is a finite extension of local or global fields, then there is a natural inclusion  $W_L \rightarrow W_K$ . Let  $H$  be a connected reductive group split over  $K$ , and  $G = \mathrm{Res}_{L/K}(H \times_K L)$ . Then there is a natural diagonal embedding

$${}^L H = \hat{H} \times W_K \rightarrow \left( \prod_{W_L \setminus W_K} \hat{H} \right) \rtimes W_K = {}^L G,$$

and the corresponding functorial lift is called base change.

# 6 Methods of Proof

## 6.1 Trace Formula

One method of proof for the functoriality conjectures is the trace formula. This subject is enormously complicated, but we can give a sketch of the motivating idea, following [Gel84].

Say we want to understand the representations of  $G(\mathbb{A}_K)$  occurring in  $L^2(G(K) \backslash G(\mathbb{A}_K))$  (or possibly the cuspidal space  $L^2_0$ , since that one actually decomposes nicely). Let  $\Pi : G(\mathbb{A}_K) \rightarrow \mathrm{GL}(L^2_0)$ . As is often the case in representation theory, it is helpful to consider the characters of our representations. Of course, automorphic representations are generally infinite-dimensional, so we can't take a naive trace. Instead, if  $f$  is a smooth (i.e. smooth on the archimedean places and locally constant on non-archimedean places) compactly supported function on  $G(\mathbb{A}_K)$ , we can show that

$$\Pi(f)v = \int_{G(\mathbb{A}_K)} f(g)\Pi(g)v dg$$

is a trace class operator, so we can take its trace.

Now we compute the trace of this operator in two ways: first, using its expression as an integral operator; and second, using the decomposition of  $L^2_0$ . The equality of these two expressions is the

“trace formula” for  $G$ . By comparing the two sides of the formula, we hope to learn something about the automorphic representations of  $G$ .

This turns out to be quite hard. But what we can do is compare the trace formulas for different groups to get results comparing their automorphic representations, e.g. functoriality.

## 6.2 Converse Theorem

Another method of proof for functoriality is converse theorems—for more discussion, see [Cog03]. We’ve seen that  $L$ -functions associated to automorphic objects have nice analytic properties, such as analytic continuation and functional equation. A “converse theorem” states that if the  $L$ -function of a representation has enough nice analytic properties, then the representation is automorphic.

For simplicity, say  $H = \mathrm{GL}_m$  and  $G = \mathrm{GL}_n$ . Let  $\pi = \otimes' \pi_v$  be an automorphic representation of  $H$ , and suppose we have an  $L$ -homomorphism  ${}^L H \rightarrow {}^L G$ . Then we get local  $L$ -homomorphisms at each place, and since local functoriality is understood for general linear groups, we can make local lifts  $\pi'_v$  of  $\pi_v$  at all places. We’d like to say  $\pi' = \otimes' \pi'_v$  is the functorial lift of  $\pi$ , but this isn’t clear because we don’t know  $\pi'$  is automorphic.

But since we have local functoriality at all places, we have equality of local  $L$ -factors and  $\varepsilon$ -factors at all places, and therefore globally as well. This allows us to establish nice analytic properties for the  $L$ -function of  $\pi'$  using the  $L$ -function of  $\pi$ , which we know to be automorphic. Then by a converse theorem we can conclude that  $\pi'$  is automorphic, and therefore a functorial lift of  $\pi$ .

Still, with all this, only very few special cases of functoriality are known. The full conjectures seem to be far off.

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